On the set of associated primes of a local cohomology module

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ABSTRACT: Assume R is a local Cohen-Macaulay ring. It is shown that $\mathrm{Ass}_R(H^l_I(R))$ is finite for any ideal I and any integer l provided $\mathrm{Ass}_R(H^2_{(x,y)}(R))$ is finite for any $x,y\in R$ and $\mathrm{Ass}_R(H^3_{(x_1,x_2,y)}(R))$ is finite for any $y\in R$ and any regular sequence $x_1,x_2\in R$. Furthermore it is shown that $\mathrm{Ass}_R(H^l_I(R))$ is always finite if $\dim(R)\leq 3$. The same statement is even true for $\dim(R)\leq 4$ if R is almost factorial.

Cohomology theory is an important part of algebraic geometry. If one considers local cohomology on an affine scheme with support in a closed subset, everything can be expressed in terms of rings, ideals and modules. More precisely, let R be a noetherian ring and I an ideal of R (determining a closed subset of $\mathrm{Spec}(R)$): In this situation one studies the local cohomology modules $H^l_I(M)$, where l is a natural number and M is any R-module. As these local cohomology modules behave well under localisation, one often restricts the above situation to the case R is a local ring.

As the structure of local cohomology modules in general seems to be quite mysterious, one tries to establish finiteness properties providing a better understanding of these modules. Finiteness properties of local cohomology modules have been studied by several authors, see for example Brodmann/Lashgari Faghani [1], Huneke/Koh [5], Huneke/Sharp [6], Lyubeznik [8] and Singh [11]. For a survey of results see Huneke [7].

Throughout this paper $(R,\mathfrak{m})$ is a local noetherian ring and I an ideal of R. We deal with the question, whether the set of associated primes of every local cohomology module $H^l_I(R)$ is finite. As local cohomology modules in general are not finitely generated, this is an interesting question. For example if R is a regular local ring containing a field then $H^l_I(R)$ (for $l\geq 1$) is finitely generated only if it vanishes. This is true, because Lyubeznik ([8], [9]) proved

$$\mathrm{injdim}(H^l_I(R))\leq \dim(\mathrm{Supp}_R(H^l_I(R)))$$

for any ideal I and any l .Now if $0\neq H^l_I(R)$ was finitely generated, we would have from [10], Theorem 18.9

$$\dim(R)=\mathrm{depth}(R)=\mathrm{injdim}(H^l_I(R))\leq \dim(\mathrm{Supp}_R(H^l_I(R)))\leq \dim(R)$$

and consequently $\mathrm{Supp}_R(H^l_I(R)) = \mathrm{Spec}(R)$ contradicting $l \geq 1$.

In [3] Grothendieck conjectured that at least $\mathrm{Hom}_R(R/I, H^l_I(R))$ is always finitely generated, but soon Hartshorne was able to present the following counterexample to Grothendieck's conjecture (see [4] for details and a proof): Let k be a field, $R = k[X,Y,Z,W]/(XY - ZW) = k[x,y,z,w]$, I the ideal $(x,z) \subseteq R$. Then $\mathrm{Hom}_R(R/I, H^2_I(R))$ is not finitely generated.

However in Hartshorne's example the ring R is not regular. Thus the question arises whether Grothendieck's conjecture is true at least in the regular case. In this context there is a theorem ([5], theorem 2.3(ii) and [8], corollary 3.5) stating that if I is an ideal of a regular ring R which contains a field and b is the maximum of the heights of all primes minimal over I then for $l > b$, $\mathrm{Hom}_R(R/I, H^l_I(R))$ is finitely generated if and only if $H^l_I(R) = 0$.

Using this theorem one can give a counterexample to Grothendieck's conjecture in the regular case, an idea which is due to Hochster:

Let k be a field of characteristic zero, $R = k[[X_1, \ldots, X_6]]$ a power series ring in six variables, I_Δ the ideal generated by the 2×2-minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$. It can be seen that I_Δ has pure height two and that $H^3_{I_\Delta}(R)$ does not vanish. Now the above theorem implies $\mathrm{Hom}_R(R/I, H^3_{I_\Delta}(R))$ is not finitely generated. But theorem 7a) shows that at least the set of associated primes of $\mathrm{Hom}_R(R/I, H^3_{I_\Delta}(R))$ (which is the same as $\mathrm{Ass}_R(H^3_{I_\Delta}(R))$) is finite.

So one may wonder if any local cohomology module has only finitely many associated primes. In [7] Huneke conjectured the following: If R is a local noetherian ring, then $\mathrm{Ass}_R(H^l_I(R))$ is finite for any I and any l. This paper deals with a weaker version of Huneke's conjecture:

Conjecture $(*)$:

If R is a local Cohen-Macaulay-ring, then $\mathrm{Ass}_R(H^l_I(R))$ is finite for any I and any l.

Our main result is:

Theorem 6:

If R is a local Cohen-Macaulay-ring, the following are equivalent:

i) $(*)$ is true for R.

ii) The following two conditions are fulfilled:

a) $\mathrm{Ass}_R(H^2_{(x,y)}(R))$ is finite for every $x, y \in R$.

b) $\mathrm{Ass}_R(H^3_{(x_1,x_2,y)}(R))$ is finite, whenever $x_1, x_2 \in R$ is a regular sequence and $y \in R$.

In Remark 2 it is shown that in the regular case condition ii) a) is always satisfied. In fact at this point we will not assume that R is regular. We only need R to be a so-called almost factorial ring, which is weaker then being factorial.

Besides this main result conjecture $(*)$ is proved in several special cases, for example in case $\dim(R) \leq 3$ or furthermore in case $\dim(R) \leq 4$ provided R is almost factorial.

Before going into the details, we remark that in the sequel we use a certain (first-quadrant cohomological) spectral-sequence, the socalled Groethendieck spectral-sequence for composed functors:

If I and J are ideals of a noetherian ring R, there is a converging spectral-sequence

$$E_2^{p,q} = H^p_I(H^q_J(M)) \Rightarrow H^{p+q}_{I+J}(M)$$

for every R-module M: This is true because Γ_J of an injective module is injective again, where $\Gamma_J(M)$ is defined as the submodule $\{m \in M | J^n \cdot m = 0 \text{ for some } n\}$ of M (for details see [12], Theorem 5.8.3).

We now start our examination of conjecture $(*)$: At least for the spot $l = \mathrm{depth}(I, R)$ there are only finitely many associated primes:

Theorem 1:

Let $(R, \mathfrak{m})$ be a noetherian local ring, M a finitely generated R-module and $I \subseteq R$ an ideal. Set $t = \mathrm{depth}(I, M)$. Then

$$\mathrm{Ass}_R(H^t_I(M)) \subseteq \mathrm{Ass}_R(\mathrm{Ext}^t_R(R/I, M))$$

and so $\mathrm{Ass}_R(H^t_I(M))$ is finite.

Proof:
Choose $\mathfrak{p} \in \mathrm{Ass}_R(H^t_I(M))$ arbitrarily. Because of $H^t_{IR_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ we must have $t = \mathrm{depth}(IR_\mathfrak{p}, M_\mathfrak{p})$ and so we may assume $\mathfrak{p} = \mathfrak{m}$. Considering the structure of $H^t_I(M)$ as a direct limit of certain Ext-modules we conclude

$$\mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Ext}^t_R(R/I^n, M)) \neq 0$$

for some $n \in \mathbb{N}$. Let $x_1, \ldots, x_t \in I$ be a regular sequence. Using well-known formulas concerning Ext we get

$$\begin{aligned} 0 \neq \mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Ext}^t_R(R/I^n, M)) &= \mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Hom}_R(R/I^n, M/(x_1^n, \ldots, x_t^n)M)) \\ &= \mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Hom}_R(R/I, M/(x_1^n, \ldots, x_t^n)M)) \\ &= \mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Ext}^t_R(R/I, M)) \quad . \end{aligned}$$

Now it follows that $\mathfrak{m} \in \mathrm{Ass}_R(\mathrm{Ext}^t_R(R/I, M))$.

A theorem established by M.P. Brodmann and A. Lashgari Faghani ([1], Proposition 2.1) states something more general: Let R be a noetherian ring, $\mathfrak{a} \subseteq R$ an ideal and M a finitely generated R-module. Furthermore, let $i \in \mathbb{N}$ be given such that $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j < i$ and let $N \subseteq H^i_{\mathfrak{a}}(M)$ be a finitely generated submodule. Then, the set $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M)/N)$ is finite.

Lemma 1:

Let R be a noetherian ring, M an R-module and I, J ideals of R with $\sqrt{I} \subseteq \sqrt{J}$. Then

$$H^l_I(M) = H^l_I(M/\Gamma_J(M))$$

for any $l \geq 1$.

Proof:
Considering the long exact Γ_I-cohomology-sequence belonging to

$$0 \longrightarrow \Gamma_J(M) \longrightarrow M \longrightarrow M/\Gamma_J(M) \longrightarrow 0 \quad ,$$

we see it suffices to show $H^l_I(\Gamma_J(M)) = 0$ for $l \geq 1$. Writing M as the union of its finitely generated submodules, we reduce to the case M itself is finitely generated, so that $\Gamma_J(M)$ is an R/J^n-module ($n \gg 0$). Consequently

$$H^l_I(\Gamma_J(M)) = H^l_{I(R/J^n)}(\Gamma_J(M)) = H^l_{(0)}(\Gamma_J(M)) = 0 \quad .$$

Theorem 1 treated the case $l = \mathrm{depth}(I, R)$, and our next theorem deals with the case $l = 1$:

Theorem 2:

Let R be a noetherian local ring, $I \subseteq R$ an ideal and M a finitely generated R-module. Then $\mathrm{Ass}_R(H^1_I(M))$ is contained in $\mathrm{Ass}_R(\mathrm{Ext}^1_R(R/I, M/\Gamma_I(M)))$ and hence is finite.

Proof:
From Lemma 1 we get

$$H^1_I(M) = H^1_I(M/\Gamma_I(M))$$

and $\Gamma_I(M/\Gamma_I(M)) = 0$ implies $\mathrm{depth}(I, M/\Gamma_I(M)) \geq 1$. So theorem 2 becomes a corollary of theorem 1.

The next theorem shows that in studying conjecture $(*)$, it suffices to examine $H^j_I(R)$ when $\mathrm{height}(I)$ equals $j-1$ or j.

Theorem 3:

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay-ring, $I \subseteq R$ an ideal, $j > \mathrm{height}(I)$ and $H^j_I(R) \neq 0$. Then there exists an ideal $\tilde{I} \supseteq I$ of height $j-1$ such that the natural homomorphism

$$H^j_{\tilde{I}}(R) \longrightarrow H^j_I(R)$$

becomes an isomorphism.

Proof:
We may assume $\mathrm{height}(I) < j-1$. Set $t = \mathrm{height}(I)$ and let $x_1, \ldots, x_t \in I$ be a regular sequence. We denote the associated primes of $R/(x_1, \ldots, x_t)$ by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, enumerated in such a way that

$$I \subseteq \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r \quad ,$$

$$I \not\subseteq \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n \quad .$$

We necessarily have $r < n$, because otherwise $\sqrt{I} = \sqrt{(x_1, \ldots, x_t)}$ and consequently $H^j_I(R) = 0$, contrary to the assumptions. Using prime avoidance we choose

$$y \in (\mathfrak{p}_{r+1} \cap \ldots \cap \mathfrak{p}_n) \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r)$$

and consider the Mayer-Vietoris-sequence with respect to the ideals $(y), I$ and the R-module $H^t_{(x_1,\ldots,x_t)}(R) =: M$:

$$\begin{aligned} H^{j-t-1}_{I\cap(y)}(M) \longrightarrow H^{j-t}_{(I,y)}(M) \longrightarrow H^{j-t}_I(M) \oplus H^{j-t}_{(y)}(M) \\ \longrightarrow H^{j-t}_{I\cap(y)}(M) \quad . \end{aligned}$$

In the sequel we write $(\underline{x})$ for the ideal $(x_1, \ldots, x_t)$ of R. Because $j - t \geq 2$ and $I \cap (y) \subseteq \sqrt{(\underline{x})}$ it follows that $H^{j-t}_{(y)} = 0$ and both the leftmost and rightmost term in this sequence vanish; so the second arrow is an isomorphism. Using the spectral-sequences for the composed functors $\Gamma_{(I,y)} \circ \Gamma_{(\underline{x})}$ and $\Gamma_I \circ \Gamma_{(\underline{x})}$ we conclude

$$\begin{aligned} H^j_{(I,y)}(R) &= H^{j-t}_{(I,y)}(M) \\ &= H^{j-t}_I(M) \\ &= H^j_I(R) \quad . \end{aligned}$$

By construction $\mathrm{height}(I, y) = \mathrm{height}(I) + 1$. Now the statement of the theorem follows inductively.

The following corollary is the first step in a series of reductions of conjecture $(*)$:

Corollary 1:

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay-ring and $j \in \mathbb{N}$. Then the following two statements are equivalent:
i) $\mathrm{Ass}_R(H^j_I(R))$ is finite for each ideal $I \subseteq R$.
ii) $\mathrm{Ass}_R(H^j_I(R))$ is finite for each ideal $I \subseteq R$ satisfying $\mathrm{height}(I) = j - 1$.

Proof:
Follows immediately from theorem 3.

Using the ideas of the proof of theorem 3 one can show that $H^j_I(R)$ has only finitely many associated primes of height j:

Corollary 2:

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay-ring, I an ideal of R and $j \in \mathbb{N}$. Then

$$\mathrm{Supp}_R(H^j_I(R)) \cap \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{height}(\mathfrak{p}) = j\}$$

is finite and therefore $H^j_I(R)$ has only finitely many associated prime ideals of height j.

Proof:
We may assume $\mathrm{height}(I) \leq j - 1$. Because of theorem 3 we may even assume that the height of I equals $j - 1$. Let $x_1, \ldots, x_{j-1} \in I$ be a regular sequence and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ the associated primes of $R/(x_1, \ldots, x_{j-1})$, enumerated in a way that we have

$$I \subseteq \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r \quad ,$$

$$I \not\subseteq \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n \quad .$$

We assume $r < n$ (if $r = n$ we have $\sqrt{I} = \sqrt{(\underline{x})}$ and therefore $H_I^j(R) = 0$). Set $J := \mathfrak{p}_{r+1} \cap \ldots \cap \mathfrak{p}_n$ and consider the following part of a Mayer-Vietoris-sequence:

$$H_{I+J}^j(R) \longrightarrow H_I^j(R) \oplus H_J^j(R) \longrightarrow H_{(x_1,\ldots,x_{j-1})}^j(R) = 0 \quad .$$

It follows $\mathrm{Supp}_R(H_I^j(R)) \subseteq \mathcal{V}(I+J)$. As $\mathrm{height}(I+J) \geq j$, there are only finitely many primes of height j in $\mathrm{Supp}_R(H_I^j(R))$.

The methods we have developed so far suffice to prove conjecture $(*)$ in case $\dim(R) \leq 3$:

Corollary 3:

Let R be a local Cohen-Macaulay-ring of dimension at most three, I an ideal of R and $j \in \mathbb{N}$. Then $H_I^j(R)$ has only finitely many associated primes.

Proof:
Case $\dim(R) = 2$: If $j = 2$, the statement follows immediately from theorems 1 and 3. The case $j = 1$ is done by theorem 2.
Case $\dim(R) = 3$: The case $j = 3$ follows at once from theorems 1 and 3. $j = 1$ is again done by theorem 2. If $j = 2$, we assume $\mathrm{height}(I) = 1$ by theorem 2. Now the statement follows from Corollary 2.

Lemma 2:

Let I be an ideal of a noetherian ring R and M any R-module. Then $\mathrm{Ass}_R(M/\Gamma_I(M)) = \mathrm{Ass}_R(M) \cap (\mathrm{Spec}(R) \backslash \mathcal{V}(I))$.

Proof:
If $\mathfrak{p}$ is associated to $M/\Gamma_I(M)$ we get from an exact sequence

$$0 \longrightarrow R/\mathfrak{p} \longrightarrow M/\Gamma_I(M)$$

an exact sequence

$$0 \longrightarrow \Gamma_I(R/\mathfrak{p}) \longrightarrow \Gamma_I(M/\Gamma_I(M)) = 0$$

and consequently $\mathfrak{p}$ does not contain I. Choose $m \in M$ satisfying $\Gamma_I(M) : m = \mathfrak{p}$. Localizing we conclude

$$0 : \frac{m}{1} = \Gamma_{IR_\mathfrak{p}}(M_\mathfrak{p}) : \frac{m}{1} = \mathfrak{p}R_\mathfrak{p} \quad .$$

From our assumptions it follows that $\frac{m}{1} \neq 0$, because otherwise there would exist $s \in R \backslash \mathfrak{p}$ with $sm = 0$, contradicting $\Gamma_I(M) : m = \mathfrak{p}$. Hence $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, equivalently $\mathfrak{p} \in \mathrm{Ass}_R(M)$.
On the other hand, if we choose $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap (\mathrm{Spec}(R) \backslash \mathcal{V}(I))$, $\mathfrak{p}$ cannot be associated to $\Gamma_I(M)$ and consequently must be associated to $M/\Gamma_I(M)$ (consider $0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$ exact).

Lemma 3:

Let I be an ideal of a local Cohen-Macaulay-ring R and set $l = \mathrm{height}(I)+1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the elements of $\{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p}$ minimal over I and $\mathrm{height}(\mathfrak{p}) = \mathrm{height}(I)\}$. Set $I^{pure} := \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$. Then finiteness of $\mathrm{Ass}_R(H^l_{I^{pure}}(R))$ implies finiteness of $\mathrm{Ass}_R(H^l_I(R))$.

Proof:
Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the elements of $\{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p}$ minimal over I and $\mathrm{height}(\mathfrak{p}) > \mathrm{height}(I)\}$ (without restriction assume $m \geq 1$) and set $I'' := \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_m$. Then $\sqrt{I} = I^{pure} \cap I''$. Consider the Mayer-Vietoris-sequence

$$H^l_{I^{pure}+I''}(R) \longrightarrow H^l_{I^{pure}}(R) \oplus H^l_{I''}(R) \longrightarrow H^l_I(R) \longrightarrow H^{l+1}_{I^{pure}+I''}(R) \quad .$$

As by construction $\mathrm{height}(I^{pure} + I'') \geq \mathrm{height}(I) + 2 = l + 1$, the leftmost term in this sequence vanishes and the rightmost term has only finitely many associated primes. Furthermore $\mathrm{height}(I'') \geq \mathrm{height}(I) + 1 = l$ and so $H^l_{I''}(R)$ has only finitely many associated prime ideals. Now the statement of the lemma is obvious.

Now we are in a position to give the next reduction of conjecture $(*)$, which roughly spoken says one may restrict to the case $j = \mu(I)$ when examining $\mathrm{Ass}_R(H^j_I(R))$:

Theorem 4:

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay-ring and $t \in \mathbb{N}$. Then the following two statements are equivalent:
i) $H^{t+1}_I(R)$ has only finitely many associated prime ideals for each ideal I of R.
ii) Whenever $x_1, \ldots, x_t \in R$ is a regular sequence and $y \in R$, the module $H^{t+1}_{(x_1,\ldots,x_t,y)}(R)$ has only finitely many associated prime ideals.

Proof:
Assume condition ii) is satisfied and let I be an arbitrary ideal of R. We have to show $\mathrm{Ass}_R(H^{t+1}_I(R))$ is finite. Using Corollary 1 we may assume $\mathrm{height}(I) = t$. Using Lemma 3 we can even assume that all primes minimal over I have height t.

Let $x_1, \ldots, x_t \in I$ be a regular sequence and denote the primes minimal over I by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. These are also minimal over $(x_1, \ldots, x_t)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the other primes minimal over $(x_1, \ldots, x_t)$ (that is, the ones not containing I). As all the ideals $\mathfrak{p}_i$ and $\mathfrak{q}_j$ have height t, we may choose a

$$y' \in (\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n) \setminus (\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_m) \quad .$$

Now a suitable power y of y' will satisfy

$$y \in I \setminus (\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_m) \quad .$$

By using Lemma 2 it follows that y is not in any prime ideal associated to the R-module $(R/(x_1^s, \ldots, x_t^s))/\Gamma_I(R/(x_1^s, \ldots, x_t^s))$ ($s \in \mathbb{N}$ arbitrary). Consequently y operates injectively on $(R/(x_1^s, \ldots, x_t^s))/\Gamma_I(R/(x_1^s, \ldots, x_t^s))$. From the exactness of the direct limit-functor we conclude, that y operates injectively on

$$\begin{aligned} &\varinjlim_{s\in\mathbb{N}}[(R/(x_1^s, \ldots, x_t^s))/\Gamma_I(R/(x_1^s, \ldots, x_t^s))] \\ =&\varinjlim_{s\in\mathbb{N}}(R/(x_1^s, \ldots, x_t^s))/\Gamma_I(\varinjlim_{s\in\mathbb{N}}(R/(x_1^s, \ldots, x_t^s))) \\ =&H^t_{(x_1,\ldots,x_t)}(R)/\Gamma_I(H^t_{(x_1,\ldots,x_t)}(R)) \quad . \end{aligned}$$

Call this property of y $(**)$. From well-known spectral-sequence-arguments it follows

$$\begin{aligned} H_I^{t+1}(R) &= H_I^1(H^t_{(x_1,\ldots,x_t)}(R)) \\ &\overset{(+)}{=} H_I^1(H^t_{(x_1,\ldots,x_t)}(R)/\Gamma_I(H^t_{(x_1,\ldots,x_t)}(R))) \\ &\overset{(**)}{=} \Gamma_I(H^1_{(y)}(H^t_{(x_1,\ldots,x_t)}(R)/\Gamma_I(H^t_{(x_1,\ldots,x_t)}(R)))) \\ &\subseteq H^1_{(y)}(H^t_{(x_1,\ldots,x_t)}(R)/\Gamma_I(H^t_{(x_1,\ldots,x_t)}(R))) \\ &\overset{(+)}{=} H^1_{(y)}(H^t_{(x_1,\ldots,x_t)}(R)) \\ &= H^{t+1}_{(x_1,\ldots,x_t,y)}(R) \quad . \end{aligned}$$

The two equalities $(+)$ follow from Lemma 1. The above inclusion finishes our proof, since we can conclude

$$|\operatorname{Ass}_R(H_I^{t+1}(R))| \leq |H^{t+1}_{(x_1,\ldots,x_t,y)}(R)| < \infty \quad .$$

Using the various statements established so far, we can prove conjecture $(*)$ in the case R is regular of dimension at most four (cf. Theorem 5); in fact we do not actually need that R is regular. We will only use the fact that every height one prime ideal is principal

up to radical; this is true if R is a Krull domain whose divisor class group is torsion (cf. [2], Proposition 6.8). Krull domains whose divisor class group is torsion are usually called almost factorial. In particular if R is factorial, it is almost factorial.

Theorem 5:

Let R be a local almost factorial Cohen-Macaulay-ring of dimension at most four, I an ideal of R and $j \in \mathbf{N}$. Then $H_I^j(R)$ has only finitely many associated primes, that is, in these cases conjecture $(*)$ is true.

Proof:
We may restrict ourselves to the case $\dim(R) = 4$. The case $j = 0$ is trivial, $j = 1$ follows from theorem 2, $j = 3$ follows from our corollaries 1 and 2 and $j = 4$ from theorem 3. In the remaining case $j = 2$ we may assume height$(I) = 1$ (theorem 3). Using Lemma 3, we may even assume that all primes minimal over I have height one. In our case this means that I is principal up to radical and so $H_I^2(R) = 0$.

Theorem 6 is our final reduction of conjecture $(*)$, allowing us to restrict ourselves to the examination of "two" special cases (for the regular case, see remark 2):

Theorem 6:

Let R be a local Cohen-Macaulay-ring. Then the following two statements are equivalent:
i) $H_I^j(R)$ has only finitely many associated prime ideals for each ideal I of R and each $j \in \mathbf{N}$.
ii) The following two conditions are satisfied:
a) $\mathrm{Ass}_R(H^2_{(x,y)}(R))$ is finite for every $x, y \in R$.
b) $\mathrm{Ass}_R(H^3_{(x_1,x_2,y)}(R))$ is finite whenever $x_1, x_2 \in R$ is a regular sequence and $y \in R$.

Proof:
We only have to show ii) implies i): We do this by induction on j:
$j = 0$: Easy.
$j = 1$: Theorem 2.
$j = 2, 3$: Theorem 4.
$j \geq 4$: Using theorem 4 we assume that $I = (x_1, \ldots, x_j)$ (for some $x_1, \ldots, x_j \in R$). Here [ ] means Gaussian brackets, that is $[q] := max\{i \in \mathbf{Z} | i \leq q\}$ for rational q. Set $I' := (x_1, \ldots, x_{[j/2]}), I'' := (x_{[j/2]+1}, \ldots, x_j) \subseteq R$ ideals and consider the following Mayer-Vietoris-sequence:

$$H^{j-1}_{I'}(R) \oplus H^{j-1}_{I''}(R) \longrightarrow H^{j-1}_{I' \cap I''}(R) \longrightarrow H^j_I(R) \longrightarrow H^j_{I'} \oplus H^j_{I''}(R) \quad .$$

Combined with our induction hypothesis (using $j-1 \geq j-([j/2]+1)+1$) we get from this an isomorphism

$$H^{j-1}_{I' \cap I''}(R) \longrightarrow H^j_I(R) \quad .$$

Another application of our induction hypothesis finishes the proof of the theorem.

Remark 1:

i) Let R be a local Cohen-Macaulay-ring, $n \in \{2,3\}$ and $x_1, \ldots, x_n \in R$. Now from $|\operatorname{Ass}_R(H^n_{(x_1,\ldots,x_n)}(R))| < \infty$ conjecture $(*)$ would follow. We can write the module $H^n_{(x_1,\ldots,x_n)}(R)$ in another way. First we have

$$H^n_{(x_1,\ldots,x_n)}(R) = H^1_{(x_1)}(H^{n-1}_{(x_2,\ldots,x_n)}(R))$$

and from the right-exactness of $H^1_{(x_1)}$ we may conclude

$$H^1_{(x_1)}(H^{n-1}_{(x_2,\ldots,x_n)}(R)) = H^1_{(x_1)}(R) \otimes_R H^{n-1}_{(x_2,\ldots,x_n)}(R) \quad .$$

An easy induction proof gives us

$$H^n_{(x_1,\ldots,x_n)}(R) = H^1_{(x_1)}(R) \otimes_R \ldots \otimes_R H^1_{(x_n)}(R) = (R_{x_1}/R) \otimes_R \ldots \otimes_R (R_{x_n}/R) \quad .$$

So for conjecture $(*)$ it is sufficient to prove

$$|\operatorname{Ass}_R((R_{x_1}/R) \otimes_R \ldots \otimes_R (R_{x_n}/R))| < \infty$$

for $n \in \{2,3\}$.

ii) Consider the complete case, that is, R is a local complete Cohen-Macaulay-ring. Similar to theorem 6, condition ii) assume $t \in \{1,2\}$, $x_1, \ldots, x_t \in R$ a regular sequence and $y \in R$. Consider R as an $R[[T]]$-module via the R-algebra homomorphism $R[[T]] \longrightarrow R$ sending T to y. We then calculate

$$\begin{aligned} H^{t+1}_{(x_1,\ldots,x_t,y)}(R) &= H^{t+1}_{(x_1,\ldots,x_t,T)}(R) \\ &= H^{t+1}_{(x_1,\ldots,x_t,T)}(R[[T]]/(T-y)) \\ &= H^{t+1}_{(x_1,\ldots,x_t,T)}(R[[T]])/(T-y)H^{t+1}_{(x_1,\ldots,x_t,T)}(R[[T]]) \quad . \end{aligned}$$

Since $x_1, \ldots, x_t, T \in R[[T]]$ is a regular sequence, it is in the complete case sufficient (for conjecture $(*)$) to show that whenever $t \in \{2,3\}$, $x_1, \ldots, x_t \in R$ is a regular sequence and $y \in R$ we have

$$|\operatorname{Ass}_R(H^t_{(x_1,\ldots,x_t)}(R)/yH^t_{(x_1,\ldots,x_t)}(R))| < \infty \quad .$$

Remark 2:

If R is an almost factorial local ring, condition a) from theorem 6 ii) is automatically fulfilled. To show this we may, with respect to theorem 3, assume height$(x,y)=1$. Using Lemma 3 we may even assume that all primes minimal over (x,y) have height one. As R is almost factorial, it follows that (x,y) is principal up to radical and so $H^2_{(x,y)}(R)=0$.

The remaining theorems 7 and 8 prove conjecture $(*)$ in certain generic cases (where R/I is Cohen-Macaulay); theorem 7 treats the equicharacteristic case and theorem 8 deals with mixed characteristics.

Theorem 7:

a) let k be a field, $R=k[[X_1,\ldots,X_6]]$ a power series ring in six indeterminates, $\Delta_1 := X_2X_6-X_3X_5, \Delta_2 := X_1X_6-X_3X_4, \Delta_3 := X_1X_5-X_2X_4$ (these are the 2×2-minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$), I the ideal $(\Delta_1,\Delta_2,\Delta_3)\subseteq R$. Then $\mathrm{Supp}_R(H^3_I(R))\subseteq \{(X_1,\ldots,X_6)\}$ and consequently $\mathrm{Ass}_R(H^3_I(R))$ is finite.
b) Let R be a local equicharacteristic Cohen-Macaulay-ring and $x_1,\ldots,x_6\in R$ be a regular sequence. Let $\delta_1 := x_2x_6-x_3x_5, \delta_2 := x_1x_6-x_3x_4, \delta_3 := x_1x_5-x_2x_4$ and I be the ideal $(\delta_1,\delta_2,\delta_3)\subseteq R$. Then $\mathrm{Ass}_R(H^3_I(R))$ is finite.

Proof:
a) It is well-known that R/I is a Cohen-Macaulay domain of dimension 4. Consequently I is a prime ideal of height two. From [10], Theorem 30.4.(ii) it follows that

$$Sing(R/(\Delta_1))\subseteq\{\mathfrak{p}\in \mathrm{Spec}(R/(\Delta_1))|\mathfrak{p}\supseteq (X_2,X_6,X_3,X_5)\} \quad .$$

Here $Sing(R/(\Delta_1))$ means the set of all primes $\mathfrak{p}$ satisfying $(R/(\Delta_1))_{\mathfrak{p}}$ is not regular. Furthermore we have

$$Sing(R/(\Delta_2))\subseteq\{\mathfrak{p}\in \mathrm{Spec}(R/(\Delta_1))|\mathfrak{p}\supseteq (X_1,X_6,X_3,X_4)\}$$

and

$$Sing(R/(\Delta_3))\subseteq\{\mathfrak{p}\in \mathrm{Spec}(R/(\Delta_1))|\mathfrak{p}\supseteq (X_1,X_5,X_2,X_4)\} \quad .$$

Choose $\mathfrak{p}\in \mathrm{Spec}(R/I)\setminus\{(X_1,\ldots,X_6)\}$ arbitrarily. We have to show $H^3_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})=0$. From our above calculations we know there is an $i\in\{1,2,3\}$ with $\mathfrak{p}\notin Sing(R/(\Delta_i))$. Thus $(R/(\Delta_i))_{\mathfrak{p}}$ is factorial. Combining this with the fact that $I/(\Delta_i)$ is a prime ideal of height one, we conclude the ideal $IR_{\mathfrak{p}}/(\Delta_i)R_{\mathfrak{p}}\subseteq R_{\mathfrak{p}}/(\Delta_i)R_{\mathfrak{p}}$ is principal. This finally shows

$$0=H^2_{IR_{\mathfrak{p}}/(\Delta_i)R_{\mathfrak{p}}}(H^1_{(\Delta_i)R_{\mathfrak{p}}}(R_{\mathfrak{p}}))=H^3_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \quad .$$

b) We may assume that R is complete, because if the statement is proved in the complete case, then the formula

$$\mathrm{Ass}_R(H^3_I(R)) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(H^3_I R(R))} \mathrm{Ass}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$$

(cf. [10], Theorem 23.2.(ii)) implies finiteness of $\mathrm{Ass}_R(H^3_I(R))$ (each $\mathrm{Ass}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$ contains a $\mathfrak{q}$ with $\mathfrak{q}\cap R = \mathfrak{p}$).
Let $k \subseteq R$ be a field, $k[[X_1,\dots,X_6]]$ be a power series ring in six variables and $\Delta_1,\Delta_2,\Delta_3 \in k[[X_1,\dots,X_6]]$ (like in a)) the 2×2-minors of $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$. The flat k-algebrahomomorphism

$$k[[X_1,\dots,X_6]] \longrightarrow R$$

with $X_i \mapsto x_i$ $(i=1,\dots,6)$ sends Δ_j to δ_j $(j=1,2,3)$. This implies

$$H^3_I(R) = H^3_{(\Delta_1,\Delta_2,\Delta_3)}(R) = H^3_{(\Delta_1,\Delta_2,\Delta_3)}(k[[X_1,\dots,X_6]]) \otimes_{k[[X_1,\dots,X_6]]} R$$

and we conclude

$$\mathrm{Ass}_R(H^3_I(R)) \subseteq \mathrm{Ass}_R(R/(X_1,\dots,X_6)R) \quad ,$$

from [10], Theorem 23.2.(ii), which finally proves b).

Theorem 8:

a) Let p be a prime number, C a complete p-ring, $R = C[[X_1,\dots,X_6]]$ a power series ring in six variables and set $\Delta_1 := X_2X_6 - X_3X_5, \Delta_2 := X_1X_6 - X_3X_4, \Delta_3 := X_1X_5 - X_2X_4$ (these are the 2×2-minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$), I the ideal $(\Delta_1,\Delta_2,\Delta_3) \subseteq R$. Then $\mathrm{Supp}_R(H^3_I(R)) \subseteq \mathcal{V}((X_1,\dots,X_6))$ and consequently $\mathrm{Ass}_R(H^3_I(R))$ is finite.
b) Let p be a prime number, $(R,\mathfrak{m})$ be a local Cohen-Macaulay-ring satisfying $char(R) = 0$, $char(R/\mathfrak{m}) = p$ and $x_1,\dots,x_6 \in R$ with the property that $p, x_1,\dots,x_6 \in R$ is a regular sequence. Set $\delta_1 := x_2x_6 - x_3x_5, \delta_2 := x_1x_6 - x_3x_4, \delta_3 := x_1x_5 - x_2x_4$ and let I be the ideal $(\delta_1,\delta_2,\delta_3) \subseteq R$. Then $\mathrm{Ass}_R(H^3_I(R))$ is finite.

Proof:
a) The proof is practically the same as the proof of theorem 7 a).
b) Like in the proof of theorem 7 b), we may assume that R is complete. According to [10], theorem 29.3 R has a coefficient ring $C \subseteq R$. Let $C[[X_1,\dots,X_6]]$ be a power series ring in six variables and $\Delta_1,\Delta_2,\Delta_3 \in C[[X_1,\dots,X_6]]$ (like in a)) the 2×2-minors of

$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$. The rest of the proof may be copied from the proof of theorem 7 b) until one finally gets

$$\mathrm{Ass}_R(H_I^3(R)) \subseteq \mathrm{Ass}_R(R/(X_1, \ldots, X_6)R) \cup \mathrm{Ass}_R(R/(p, X_1, \ldots, X_6)R) \quad ,$$

which proves b).

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